

TRANSPORT NETWORKS REVISITED: WHY DUAL GRAPHS?

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Abstract

Deterministic equilibrium flows in transport networks can be investigated by means of Markov's processes defined on the dual graph representations of the network. Sustained movement patterns are generated by a subset of automorphisms of the graph spanning the spatial network of a city naturally interpreted as random walks. Random walks assign absolute scores to all nodes of a graph and embed space syntax into Euclidean space.

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1 Motivation

In most of researches devoted to the improving of transport routes, the optimization of power grids, and the pedestrian movement surveys the relationships between certain components of the urban texture are often measured along streets and routes considered as edges of a planar graph, while the traffic end points and street junctions are treated as nodes. Such a *primary* graph representation of urban networks is grounded on relations between junctions through the segments of streets. A usual city map based on Euclidean geometry can be considered as an example of primary city graphs.

On the contrary, in another graph representation of urban networks known as *dual*, relations between streets are encoded through their junctions. Streets within that are treated as nodes while their intersections are considered as edges.

Dual city graphs are extensively investigated within the concept of *space syntax*, a theory developed in the late 1970s, that seeks to reveal the mutual effects of complex spatial urban networks on society and vice versa, [1, 2].

Recently, while acting as a referee in *Physica A*, we have been asked to review a manuscript devoted to city space syntax in which the dual graph representation of urban patterns is used. Among the technical remarks expressed by the previous referees, we have discovered the following criticism:

"[The] dual option in representing cities would have deserved a deeper discussion, as it must be said that no one in the world, space syntax apart, appears to use it ... space syntax supporters and scientists should dedicate more efforts to the legitimation of their theory's fundamental notions in the universally accepted language of the wider scientific community."

Being not obliged to reply personally, we nevertheless must answer such a corporative challenge by explaining the deep arguments in favor of studying namely the dual graph representations.

Legitimacy of the space syntax approach is based on the surprisingly high degree of correlation between aggregate human movement rates and spatial configuration analyzed in space syntax theory by means of dual city graphs. We recall here that the key result of space syntax research is that the pattern of spatial integration in the urban grid is a key determinant of pedestrian movement in cities across the world, [3].

We emphasize that the dual graph analysis neither incorporates many of the factors considered important in previous models of human behavior in urban environments (such as the motivations and the origin-destination information) nor direct account was taken of the metric properties of space (lengths of streets), [4]. Nevertheless, the robustness of agreement between space syntax predictions and rush hour movement rates is now supported by a number of similar studies of pedestrian movement in different parts of the world and in an everyday commercial work of the *Space Syntax Ltd.*, [5]. Similar results also exist for vehicular movement [6] showing that the spatial configuration of the urban grid is by itself a consistent factor driving transport flows.

The high quality agreement with the empirical data obtained from pedestrian movement surveys looks amazing even for those who invented the space syntax approach. Quoting [4], there is nothing else to evoke but that

"the way people understand their environment and decide on movement behaviors is somehow implicitly embedded in space syntax analysis".

In the next section, we address three questions of essential importance for the analysis of transport networks:

1. Why does the space syntax approach based on the pure configurational analysis of dual graphs appear to be so successful in prediction of pedestrian and vehicular flows, essentially in rush hours?

2. Why does the measurement of the simultaneous importance of spaces in a city network by a quantity which is nothing else, but an element of a transition probability matrix determining a Markov chain fit best the empirical data on crowding patterns?
3. What is the geometrical structure beyond space syntax?

We show that equilibrium transport flows through edges of a connected undirected graph G at equilibrium are related to the stationary distribution of random walks defined on its dual counterpart G^* . Random walks constitute a fundamental process defined on a graph generated by the set of its automorphisms preserving the notion of connectivity which assign the absolute scores to all nodes of the graphs. Random walks embed the space syntax of any transport network into the Euclidean space \mathbb{R}^{N-1} in which distances and angles get clear statistical interpretations.

2 Mathematics of transport networks

Any graph representation naturally arises as the outcome of a categorization, when we abstract a real world system by eliminating all but one of its features and by grouping together things (or places) sharing a common attribute. For instance, the common attribute of all open spaces in city space syntax is that we can move through them. All elements called nodes that fall into one and the same group V are considered as essentially identical; permutations of them within the group are of no consequence. The symmetric group \mathbb{S}_N consisting of all permutations of N elements (N being the cardinality of the set V) constitute therefore the symmetry group of V . If we denote by $E \subseteq V \times V$ the set of ordered pairs of nodes called edges, then a graph is a map $G(V, E) : E \rightarrow K \subseteq \mathbb{R}_+$ (we suppose that the graph has no multiple edges). If two nodes are adjacent, $(i, j) \in E$ we write $i \sim j$.

2.1 Why dual graphs?

First, we establish a connection between transport flows on the graph G and random walks on its dual counterpart G^* .

Given a connected undirected graph $G(V, E)$, in which V is the set of nodes and E is the set of edges, we introduce the traffic function $f : E \rightarrow (0, \infty[$ through every edge $e \in E$. It then follows from the Perron-Frobenius theorem [7] that the linear equation

$$f(e) = \sum_{e' \sim e} f(e') \exp(-h \ell(e')), \quad (1)$$

where the sum is taken over all edges $e' \in E$ which have a common node with e , has a unique positive solution $f(e) > 0$, for every edge $e \in E$, for a fixed positive constant $h > 0$ and a chosen set of positive *metric length* distances $\ell(e) > 0$. This solution is naturally identified with the traffic equilibrium state of the

transport network defined on G , in which the permeability of edges depends upon their lengths. The parameter h is called the volume entropy of the graph G , while the volume of G is defined as the sum

$$\text{Vol}(G) = \frac{1}{2} \sum_{e \in E} \ell(e).$$

The volume entropy h is defined to be the exponential growth of the balls in a universal covering tree of G with the lifted metric, [8]-[11].

The degree of a node $i \in V$ is the number of its neighbors in G , $\deg_G(i) = k_i$. It has been shown in [11] that among all undirected connected graphs of normalized volume, $\text{Vol}(G) = 1$, which are not cycles and for which $k_i \neq 1$ for all nodes, the minimal value of the volume entropy, $\min(h) = \frac{1}{2} \sum_{i \in V} k_i \log(k_i - 1)$ is attained for the length distances

$$\ell(e) = \frac{\log((k_i - 1)(k_j - 1))}{2 \min(h)}, \quad (2)$$

where k_i and k_j are the degrees of the nodes linked by $e \in E$. It is then obvious that substituting (2) and $\min(h)$ into (1) the operator $\exp(-h\ell(e'))$ is given by a symmetric Markov transition operator,

$$f(e) = \sum_{e' \sim e} \frac{f(e')}{\sqrt{(k_i - 1)(k_j - 1)}}, \quad (3)$$

where i and j are the nodes linked by $e' \in E$, and the sum in (3) is taken over all edges $e' \in E$ which share a node with e . The symmetric operator (3) rather describes time reversible random walks over edges than over nodes. In other words, we are invited to consider random walks described by the symmetric operator defined on the dual graph G^* .

The Markov process (3) represents the conservation of the traffic volume through the transport network, while other solutions of (1) are related to the possible termination of travels along edges. If we denote the number of neighbor edges the edge $e \in E$ has in the dual graph G^* as $q_e = \deg_{G^*}(e)$, then the simple substitution shows that $w(e) = \sqrt{q_e}$ defines an eigenvector of the symmetric Markov transition operator defined over the edges E with eigenvalue 1. This eigenvector is positive and being properly normalized determines the relative traffic volume through $e \in E$ at equilibrium.

Eq.(3) shows the essential role Markov's chains defined on edges play in equilibrium traffic modelling and emphasizes that the degrees of nodes are a key determinant of the transport networks properties.

The notion of traffic equilibrium had been introduced by J.G. Wardrop in [12] and then generalized in [13] to a fundamental concept of network equilibrium. Wardrop's traffic equilibrium is strongly tied to the human apprehension of space since it is required that all travellers have enough knowledge of the transport network they use. The human perception of places is not an entirely Euclidean one, but are rather related to the perceiving of the vista spaces (viewable spaces

of streets and squares) as single units and to the understanding of the topological relationships between these vista spaces, [14].

The use of Eq.(3) also helps to clarify the inconsistency of the traditional axial technique widely implemented in space syntax theory. Lines of sight are oversensitive to small deformations of the grid, which leads to noticeably different axial graphs for systems that should have similar configuration properties. Long straight paths, represented by single lines, appear to be overvalued compared to curved or sinuous paths as they are broken into a number of axial lines that creates an artificial differentiation between straight and curved or sinuous paths that have the same importance in the system [15]. Eq.(3) shows that the nodes of a dual graph representing the open spaces in the spatial network of an urban environment should have an individual meaning being an entity characterized by the certain traffic volume capacity.

Decomposition of city space into a complete set of intersecting open spaces characterized by the traffic volume capacities produces a spatial network which we call the *dual* graph representation of a city.

2.2 Why random walks?

While analyzing a graph, whether it is primary or dual, we assign the absolute scores to all nodes based on their properties with respect to a transport process defined on that. Indeed, the nodes of $G(V, E)$ can be weighted with respect to some measure $m = \sum_{i \in V} m_i \delta_{ij}$, specified by a set of positive numbers $m_i > 0$. The space $\ell^2(m)$ of square-summable functions with respect to the measure m is a Hilbert space $\mathcal{H}(V)$.

Among all measures which can be defined on V , the set of normalized measures (or *densities*),

$$1 = \sum_{i \in V} \pi_i \delta_{ij}, \quad (4)$$

are of essential interest since they express the conservation of a quantity, and therefore may be relevant to a physical process.

The fundamental physical process defined on the graph is generated by the subset of its automorphisms preserving the notion of connectivity of nodes. An automorphism is a mapping of the object to itself preserving all of its structure. The set of all automorphisms of a graph forms a group, called the automorphism group. For each graph $G(V, E)$, there exists a unique, up to permutations of rows and columns, adjacency matrix \mathbf{A} , the $N \times N$ matrix defined by $A_{ij} = 1$ if $i \sim j$, and $A_{ij} = 0$ otherwise. As usual \mathbf{A} is identified with a linear endomorphism of $C_0(G)$, the vector space of all functions from V into \mathbb{R} . The degree of a node $i \in V$ is therefore equal to

$$k_i = \sum_{i \sim j} A_{ij}. \quad (5)$$

Let us consider the set of all linear transformations defined on the adjacency

matrix,

$$Z(\mathbf{A})_{ij} = \sum_{s,l=1}^N \mathcal{F}_{ijsl} A_{sl}, \quad \mathcal{F}_{ijsl} \in \mathbb{R}, \quad (6)$$

generated by the subset of automorphisms of the graph G .

The graph automorphisms are specified by the symmetric group \mathbb{S}_N including all admissible permutations $p \in \mathbb{S}_N$ taking $i \in V$ to $p(i) \in V$. The representation of \mathbb{S}_N consists of all $N \times N$ matrices $\mathbf{\Pi}_p$, such that $(\mathbf{\Pi}_p)_{i,p(i)} = 1$, and $(\mathbf{\Pi}_p)_{i,j} = 0$ if $j \neq p(i)$.

The function $Z(\mathbf{A})_{ij}$ should satisfy

$$\mathbf{\Pi}_p^\top Z(\mathbf{A}) \mathbf{\Pi}_p = Z(\mathbf{\Pi}_p^\top \mathbf{A} \mathbf{\Pi}_p), \quad (7)$$

for any $p \in \mathbb{S}_N$, and therefore entries of the tensor \mathcal{F} must have the following symmetry property,

$$\mathcal{F}_{p(i)p(j)p(s)p(l)} = \mathcal{F}_{ijsl}, \quad (8)$$

for any $p \in \mathbb{S}_N$. Since the action of the symmetric group \mathbb{S}_N preserves the conjugate classes of index partition structures, any appropriate tensor \mathcal{F} satisfying (8) can be expressed as a linear combination of the following tensors: $\{1, \delta_{ij}, \delta_{is}, \delta_{il}, \delta_{js}, \delta_{jl}, \delta_{sl}, \delta_{ij}\delta_{js}, \delta_{js}\delta_{sl}, \delta_{sl}\delta_{li}, \delta_{li}\delta_{ij}, \delta_{ij}\delta_{sl}, \delta_{is}\delta_{jl}, \delta_{il}\delta_{js}, \delta_{ij}\delta_{il}\delta_{is}\}$. Given a simple, undirected graph G such that $A_{ii} = 0$ for any $i \in V$ then by substituting the above tensors into (6) and taking account on symmetries we conclude that any arbitrary linear permutation invariant function must be of the form

$$Z(\mathbf{A})_{ij} = a_1 + \delta_{ij} (a_2 + a_3 k_j) + a_4 A_{ij}, \quad (9)$$

with $k_j = \deg_G(j)$ and $a_{1,2,3,4}$ arbitrary constants.

If we require that the linear function Z preserves the notion of connectivity,

$$k_i = \sum_{j \in V} Z(\mathbf{A})_{ij}, \quad (10)$$

it is clear that we should take $a_1 = a_2 = 0$ (indeed, the contributions $a_1 N$ and a_2 are incompatible with (10)) and then obtain the relation for the remaining constants, $1 - a_3 = a_4$. Introducing the new parameter $\beta \equiv a_4 > 0$, we write (9) as follows,

$$Z(\mathbf{A})_{ij} = (1 - \beta) \delta_{ij} k_j + \beta A_{ij}. \quad (11)$$

If we express (10) in the form of the probability conservation relation, then the function $Z(\mathbf{A})$ acquires a probabilistic interpretation. Substituting (11) back into (10), we obtain

$$\begin{aligned} 1 &= \sum_{j \in V} \frac{Z(\mathbf{A})_{ij}}{k_i} \\ &= \sum_{j \in V} (1 - \beta) \delta_{ij} + \beta \frac{A_{ij}}{k_i} \\ &= \sum_{j \in V} T_{ij}^{(\beta)}. \end{aligned} \quad (12)$$

The operator $T_{ij}^{(\beta)}$ represents the generalized random walk transition operator if $0 < \beta \leq k_{\max}^{-1}$ where k_{\max} is the maximal node degree in the graph G . In the random walks defined by $T_{ij}^{(\beta)}$, a random walker stays in the initial vertex with probability $1 - \beta$, while it moves to another node randomly chosen among its nearest neighbors with probability β/k_i . If we take $\beta = 1$, then the operator $T_{ij}^{(\beta)}$ describes the usual random walks discussed extensively in the classical surveys [16]-[18].

Being defined on a connected aperiodic graph, the matrix $T_{ij}^{(\beta)}$ is a real positive stochastic matrix, and therefore, in accordance to the Perron-Frobenius theorem [7], its maximal eigenvalue is 1, and it is simple. A left eigenvector

$$\pi T^{(\beta)} = \pi \quad (13)$$

associated with the eigenvalue 1 has positive entries satisfying (4). It is interpreted as a unique equilibrium state π (stationary distribution of the random walk). For any density $\sigma \in \mathcal{H}(V)$,

$$\pi = \lim_{t \rightarrow \infty} \sigma T^t. \quad (14)$$

2.3 Space syntax as Euclidean space

Markov's operators on Hilbert space appear therefore as the natural language of complex network theory and space syntax theory in particular. Now we demonstrate that random walks embed connected undirected graphs into Euclidean space, in which distances and angles acquire the clear statistical interpretations.

The Markov operator \hat{T} is *self-adjoint* with respect to the normalized measure (4) associated to the stationary distribution of random walks π ,

$$\hat{T} = \frac{1}{2} \left(\pi^{1/2} T \pi^{-1/2} + \pi^{-1/2} T^\top \pi^{1/2} \right), \quad (15)$$

where T^\top is the transposed operator.

In the theory of random walks defined on graphs [16, 18] and in spectral graph theory [19], basic properties of graphs are studied in connection with the eigenvalues and eigenvectors of self-adjoint operators defined on them. The orthonormal ordered set of real eigenvectors ψ_i , $i = 1 \dots N$, of the symmetric operator \hat{T} defines a basis in $\mathcal{H}(V)$.

In particular, the symmetric transition operator \hat{T} of the random walk defined on connected undirected graphs is

$$\hat{T}_{ij} = \begin{cases} \frac{1}{\sqrt{k_i k_j}}, & i \sim j, \\ 0, & i \not\sim j. \end{cases} \quad (16)$$

Its first eigenvector ψ_1 belonging to the largest eigenvalue $\mu_1 = 1$,

$$\psi_1 \hat{T} = \psi_1, \quad \psi_{1,i}^2 = \pi_i, \quad (17)$$

describes the *local* property of nodes (connectivity), since the stationary distribution of random walks is

$$\pi_i = \frac{k_i}{2M} \quad (18)$$

where $2M = \sum_{i \in V} k_i$. The remaining eigenvectors, $\{\psi_s\}_{s=2}^N$, belonging to the eigenvalues $1 > \mu_2 \geq \dots \mu_N \geq -1$ describe the *global* connectedness of the graph. For example, the eigenvector corresponding to the second eigenvalue μ_2 is used in spectral bisection of graphs. It is called the Fiedler vector if related to the Laplacian matrix of a graph [19].

Markov's symmetric transition operator \hat{T} defines a projection of any density $\sigma \in \mathcal{H}(V)$ on the eigenvector ψ_1 of the stationary distribution π ,

$$\sigma \hat{T} = \psi_1 + \sigma_{\perp} \hat{T}, \quad \sigma_{\perp} = \sigma - \psi_1, \quad (19)$$

in which σ_{\perp} is the vector belonging to the orthogonal complement of ψ_1 . In space syntax, we are interested in a comparison between the densities with respect to random walks defined on the graph G . Since all components $\psi_{1,i} > 0$, it is convenient to rescale the density σ by dividing its components by the components of ψ_1 ,

$$\tilde{\sigma}_i = \frac{\sigma_i}{\psi_{1,i}}. \quad (20)$$

Thus, it is clear that any two rescaled densities $\tilde{\sigma}, \tilde{\rho} \in \mathcal{H}$ differ with respect to random walks only by their dynamical components,

$$(\tilde{\sigma} - \tilde{\rho}) \hat{T}^t = (\tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp}) \hat{T}^t,$$

for all $t > 0$. Therefore, we can define the distance $\|\dots\|_T$ between any two densities established by random walks by

$$\|\sigma - \rho\|_T^2 = \sum_{t \geq 0} \left\langle \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} \left| \hat{T}^t \right| \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} \right\rangle. \quad (21)$$

or, using the spectral representation of \hat{T} ,

$$\begin{aligned} \|\sigma - \rho\|_T^2 &= \sum_{t \geq 0} \sum_{s=2}^N \mu_s^t \langle \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} | \psi_s \rangle \langle \psi_s | \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} \rangle \\ &= \sum_{s=2}^N \frac{\langle \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} | \psi_s \rangle \langle \psi_s | \tilde{\sigma}_{\perp} - \tilde{\rho}_{\perp} \rangle}{1 - \mu_s}, \end{aligned} \quad (22)$$

where we have used Diracs bra-ket notations especially convenient for working with inner products and rank-one operators in Hilbert space.

If we introduce a new inner product for densities $\sigma, \rho \in \mathcal{H}(V)$ by

$$(\sigma, \rho)_T = \sum_{t \geq 0} \sum_{s=2}^N \frac{\langle \tilde{\sigma}_{\perp} | \psi_s \rangle \langle \psi_s | \tilde{\rho}_{\perp} \rangle}{1 - \mu_s}, \quad (23)$$

then (22) is nothing else but

$$\|\sigma - \rho\|_T^2 = \|\sigma\|_T^2 + \|\rho\|_T^2 - 2(\sigma, \rho)_T, \quad (24)$$

where

$$\|\sigma\|_T^2 = \sum_{s=2}^N \frac{\langle \tilde{\sigma}_\perp | \psi_s \rangle \langle \psi_s | \tilde{\sigma}_\perp \rangle}{1 - \mu_s} \quad (25)$$

is the square of the norm of $\sigma \in \mathcal{H}(V)$ with respect to random walks defined on the graph G .

We finish the description of the $(N-1)$ -dimensional Euclidean space structure of G induced by random walks by mentioning that given two densities $\sigma, \rho \in \mathcal{H}(V)$, the angle between them can be introduced in the standard way,

$$\cos \angle(\rho, \sigma) = \frac{(\sigma, \rho)_T}{\|\sigma\|_T \|\rho\|_T}. \quad (26)$$

Random walks embed connected undirected graphs into the Euclidean space \mathbb{R}^{N-1} . This embedding can be used in order to compare nodes and to construct the optimal coarse-graining representations.

Namely, in accordance to (25), the density δ_i , which equals 1 at $i \in V$ and zero otherwise, acquires the norm $\|\delta_i\|_T$ associated to random walks defined on G . In the theory of random walks [16], its square,

$$\|\delta_i\|_T^2 = \frac{1}{\pi_i} \sum_{s=2}^N \frac{\psi_{s,i}^2}{1 - \mu_s}, \quad (27)$$

gets a clear probabilistic interpretation expressing the *access time* to a target node quantifying the expected number of steps required for a random walker to reach the node $i \in V$ starting from an arbitrary node chosen randomly among all other nodes with respect to the stationary distribution π .

The Euclidean distance between any two nodes of the graph G calculated as the distance (22) between the densities δ_i and δ_j induced by random walks,

$$K_{i,j} = \|\delta_i - \delta_j\|_T^2, \quad (28)$$

quantifies the *commute time* in theory of random walks being equal to the expected number of steps required for a random walker starting at $i \in V$ to visit $j \in V$ and then to return back to i , [16].

It is important to mention that the cosine of an angle calculated in accordance to (26) has the structure of Pearson's coefficient of linear correlations that reveals its natural statistical interpretation. Correlation properties of flows of random walkers passing by different paths have been remained beyond the scope of previous studies devoted to complex networks and random walks on graphs. The notion of angle between any two nodes of the graph arises naturally as soon as we become interested in the strength and direction of a linear relationship between two random variables, the flows of random walks moving through them. If the cosine of an angle (26) is 1 (zero angles), there is an increasing linear relationship between the flows of random walks through both nodes. Otherwise, if it is close to -1 (π angle), there is a decreasing linear relationship. The correlation is 0 ($\pi/2$ angle) if the variables are linearly independent. It is important to mention that as usual the correlation between nodes does not necessarily imply a direct causal relationship (an immediate connection) between them.

3 Discussion and Conclusion

In his speech to the meeting in the House of Lords, October 28, 1943, Sir Winston Churchill had requested that the House of Commons bombed-out on the night of May 10, 1941, be rebuilt exactly as before. "*We shape our buildings, and afterwards our buildings shape us,*" he said.

A belief in the influence of the built environment on humans was common in architectural and urban thinking for centuries. There is a tied connection between physical activity of humans, their mobility and the layout of buildings, roads, and other structures that physically define a community [20]. Spatial organization of a place has an extremely important effect on the way people move through spaces and meet other people by chance [21]. It has been also shown in [24] that space structure and its impact on movement are critical to the link between the built environment and its social functioning. In particular, the reduce of movement in a spatial pattern is crucial for the decline of new housing areas. It is well known that the urban layout effects on the spatial distribution of crime [22] and poverty [23].

Equilibrium flows in transport networks can be investigated by means of Markov's processes defined on the dual graph representations of the network. Sustained movement patterns are generated by a subset of automorphisms of the graph spanning the spatial network of a city. This process is naturally interpreted as random walks. Random walks assign absolute scores to all open spaces in the city accordingly to the quality of paths they provide to random walkers and embed city space syntax into Euclidean space.

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